

First-order circuits

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Circuits with a single *reactive* element (inductor or capacitor).

1. Capacitor

Recall that “capacitor’s law” is

$$i_C(t) = C \frac{dV_C(t)}{dt}, \text{ or} \quad (1)$$

$$v_C(t) = \frac{1}{C} \int_0^t i_C(x) dx + v_C(0^-). \quad (2)$$



Remember that the x in the integral is a dummy variable and disappears during integration. Time shows up in the integral’s upper limit, so we can’t **also** legally use t as the variable of integration.

2. Inductor

Recall that “inductor’s law” is

$$v_L(t) = L \frac{di_L(t)}{dt}, \text{ or} \quad (3)$$

$$i_L(t) = \frac{1}{L} \int_0^t v_L(x) dx + i_L(0^-). \quad (4)$$



Remember that the $t = 0$ in the lower limit and the initial condition is just a *placeholder* that sets the boundary between “then” and “for any time afterwards”. The initial condition doesn’t **need** to be at 0 seconds.

3. Single time-constant solution

a.k.a. a **first-order circuit**

Single time constant refers to the τ in the $\exp(-t/\tau)$ term. First-order refers to the circuit solution being a first order differential equation. It is always true that a circuit with N total inductors and capacitors yields an N^{th} -order differential equation.^[1]

A circuit with a reactive element (i.e. a capacitor) has a circuit analysis solution that involves a differential equation. Since there is only one reactance, there is only one time constant. The solution to this **first-order** differential equation is an exponential: $A \exp(-t/\tau) + B$



The **R** in the $\tau = R \cdot C$ is an **effective** resistance and may NOT be the value of a single physical resistor! Think "Thevenin-equivalent resistance seen by the capacitor" instead.

The general solution from your *differential equations course* is correct, but you always then need to solve for the two constants using the initial conditions of the particular circuit. A more practical and easier form is:

The only first-order differential equation solution you need to know

$$v(t) = \underbrace{\left(V_{\text{final}} - V_{\text{initial}} \right)}_{\text{change}} \underbrace{\left[1 - \exp \left(\frac{-t}{\tau} \right) \right]}_{\text{diff eq magic}} + \underbrace{V_{\text{initial}}}_{\text{don't forget}} \quad (5)$$

- $\tau = RC$ or L/R
- $t_{\text{initial}} = 0$
- replace every V with I for the current solution

If t is in seconds, then this forces τ to be in units of seconds because it makes no sense to raise a number to a power having units, for example “2.7 to the 3 meters power”.

Does it seem weird that *resistance* $\times$ *capacitance equals time*? See how the units work out once you expand them into their more basic units:

RC units

$$\begin{aligned} RC &\rightarrow \Omega \cdot F \\ \frac{V}{A} \cdot \frac{C}{V} &= \frac{C}{A} = \frac{A \cdot s}{A} \\ RC &\rightarrow \text{seconds} \end{aligned}$$

L/R units

$$\begin{aligned} \frac{L}{R} &\rightarrow \frac{H}{\Omega} \\ \frac{Wb}{A} \cdot \frac{A}{V} &= \frac{V \cdot s}{A} \cdot \frac{A}{V} \\ \frac{L}{R} &\rightarrow \text{seconds} \end{aligned}$$



The **ampere** is an SI base unit while the **coulomb** is a derived unit (A·s). But it is common to think of the reverse: an ampere is a coulomb per second.

1. ...and compels you to love Laplace transforms more than differential equations. Note that “more than” doesn’t necessarily mean you love either very much :)