

Root Mean Square quantities

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One quantity to rule them all

RMS is *everywhere*

While this discussion will be firmly set in the context of electronic circuits with currents and voltages, root mean squared is a common quantity in other domains also where a single number is useful to describe how “big” a signal is.

1. Effective values

Consider a DC voltage source V_X attached in parallel with a resistor R_{load} . The power absorbed by this resistor is then:

$$P_{\text{load}} = V_X I_X = V_X \frac{V_X}{R_{\text{load}}} = \frac{V_X^2}{R_{\text{load}}} \quad (1)$$

Now imagine that this voltage source varies with time and is periodic with period T . It makes no difference what the shape of this waveform is, just that it repeats eventually.

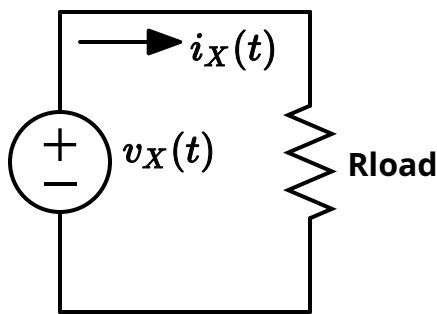


Figure 1. Simple circuit for computing power

Definition

The effective voltage V_{eff} is the value of a constant (DC) source that supplies the **same average power** to the resistor R_{load} as the signal source $v_X(t)$.

Compute the average power dissipated in the load resistor over a period of an arbitrary waveform $v_X(t)$.

$$P_R = \frac{1}{T} \int_0^T p(t) dt \quad (2)$$

$$= \frac{1}{T} \int_0^T v_X(t) i_X(t) dt \quad (3)$$

$$= \frac{1}{T} \int_0^T v_X(t) \left(\frac{v_X(t)}{R} \right) dt \quad (4)$$

$$= \frac{1}{R} \frac{1}{T} \int_0^T v_X^2(t) dt \quad (5)$$

A DC source of the correct value, the **effective voltage** will supply the same power to this resistor.

$$\frac{V_{\text{eff}}^2}{R} = P_R = \frac{1}{R} \frac{1}{T} \int_0^T v_X^2(t) dt \quad (6)$$

$$V_{\text{eff}}^2 = \frac{1}{T} \int_0^T v_X^2(t) dt \quad (7)$$

Throw in a square-root and out pops our definition of root-mean-square:^[1]

Important observations

- An RMS quantity is a **scalar value** — *not* a function of time.
- RMS is synonymous with **effective value**.
- RMS has to do with **power** at the core of its definition.

$$V_{\text{RMS}} = \sqrt{\frac{1}{T} \int_0^T v_X^2(t) dt}$$

Root
Mean
Square

Figure 2. Because S-M-R would be weird

2. Composite waveforms

Superposition is a powerful property of linear systems because it allows decomposing a signal into a sum of several signals (usually sinusoids but not necessary), which can then be treated separately.

- How does superposition work for *RMS* quantities?

Be careful, it may not be what you would guess! Power quantities behave differently than voltage and current quantities.

Consider a signal composed of two periodic waveforms v_1 and v_2 with periods T_1 and T_2 . Again, it doesn't matter the shapes as long as each part is periodic, which is the whole point of the *effective value* idea. Since both are periodic, then the composite signal is *also* periodic with period $T = T_1 T_2$. Note that T can be any **common multiple** of the individual periods.

$$v_{\text{total}} = v_1(t) + v_2(t) \quad (8)$$

Compute the (square of the) RMS of this signal by using the definition

$$V_{\text{RMS}}^2 = \frac{1}{T} \int_0^T [v_1(t) + v_2(t)]^2 dt \quad (9)$$

$$= \frac{1}{T} \int_0^T [v_1^2(t) + 2v_1(t)v_2(t) + v_2^2(t)] dt \quad (10)$$

Since integration is a **linear operator**, expand into three parts to better inspect each term

$$\begin{aligned} V_{\text{RMS}}^2 &= \frac{1}{T} \int_0^T v_1^2(t) dt \\ &\quad + \frac{2}{T} \int_0^T v_1(t)v_2(t) dt \\ &\quad + \frac{1}{T} \int_0^T v_2^2(t) dt \end{aligned} \quad (11)$$

We see familiar expressions in the first and third terms -- they are simply the RMS values of the two component signals, $(v_1 \text{ RMS})^2$ and $(v_2 \text{ RMS})^2$. The second term requires the careful treatment.

- If signals v_1 and v_2 are **orthogonal** over the interval $[0, T]$ then this term is zero (the very definition of *orthogonal*). It turns out that **any** two sinusoids of different frequencies, integrated over their *least common multiple* of their respective periods, are orthogonal.
- Another way to say this is this second term goes to zero if v_1 and v_2 are **uncorrelated**.

Therefore, for the sum of **uncorrelated** signals, the composite's RMS value is

$$V_{\text{total RMS}} = \sqrt{v_{1\text{RMS}}^2 + v_{2\text{RMS}}^2 + v_{3\text{RMS}}^2 + \dots} \quad (12)$$

This is called the **root sum (of the) squares** or RSS for short.



To repeat: RSS, root sum of the squares, **only** works when the signals are (choose one, choose all):

- ☐ uncorrelated
- ☐ orthogonal

3. RMS of common wave shapes

Few people relish the thought of computing an integral just to get an “effective value,” but that’s the definition.

Fortunately, there are a few common wave shapes that that we encounter all the time.

3.1. sinusoid

Signal: $v_X(t) = A \sin(2\pi ft)$

$$V_{\text{RMS}}(\sin) = \sqrt{\frac{1}{T} \int_0^T (A \sin(2\pi ft))^2 dt} \quad (13)$$

$$= \sqrt{\frac{1}{T} \int_0^T \frac{A^2}{2} (1 - \cos(2 \cdot 2\pi ft)) dt} \quad (14)$$

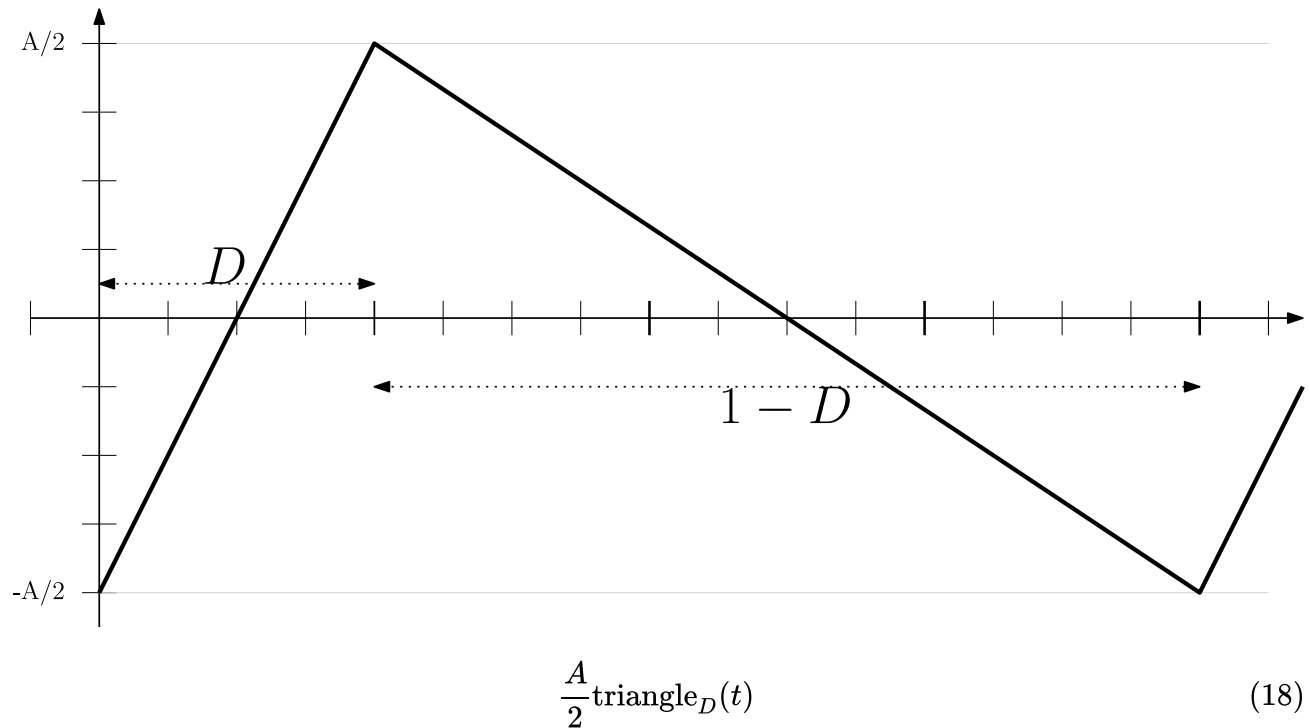
$$= \sqrt{\frac{A^2}{2T} T - \int \dots} \quad (15)$$

$$= \sqrt{\frac{A^2}{2} - 0} \quad (16)$$

$$= \frac{A}{\sqrt{2}} \quad (17)$$

- Repeat this calculation if the signal is: $v_X(t) = A \cos(2\pi ft)$?
- If you have a 120 V_{RMS} signal (say, from a power outlet), what is the peak voltage value?

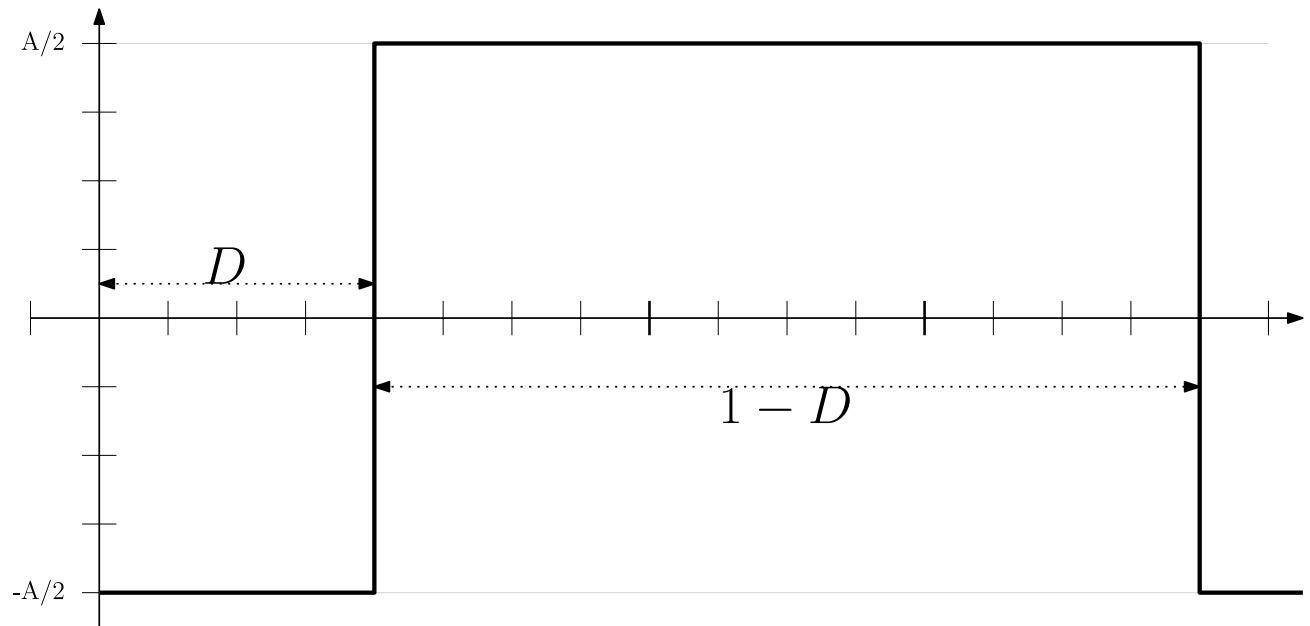
3.2. triangle



This is a fun one! Set up a triangle wave that is centered around zero, meaning that the average value must be zero. It is useful to specify the “duty cycle” as the proportion of time the waveform spends linearly rising compared to the period (the other portion the signal is linearly falling). Then you only complete this integral *once* and get a parameterized result.

You need to do this by hand at least once in your life before you memorize the result. It is useful for switching circuits with integrator-like devices (inductor current with a switched voltage, capacitor voltage with a switched current) that show up in e.g. power electronics.

3.3. square



$$\frac{A}{2} \text{square}_D(t)$$

(19)

Two fun ones, the first is easy.

- ☐ 50% duty cycle, centered around 0 with peaks at $\pm A$.
- ☐ $D\%$ duty cycle, with peaks at $\pm A$. (use D as 0..1)

3.4. Three sinusoids

A superposition harmonic sinusoids. Be sure to start with the definition of RMS and take care of your algebra. Average over the least common multiple of the three periods.

$$s(t) = A_1 \sin(2\pi 1f t) + A_3 \sin(2\pi 3f t) + A_5 \sin(2\pi 5f t) \quad (20)$$

3.5. Triangle plus square

As defined in the previous two problems.

$$s(t) = \text{square}_D(t) + \text{triangle}_D(t) \quad (21)$$

- Then compare the direct RMS evaluation result with adding the individual RMS values together.

4. Other usage domain examples

Domain	Quantity	Description
Electrical power	RMS line voltage/current	Used to specify AC power systems, equipment ratings, conductor heating, and insulation requirements. Nominal values such as 120 V AC refer to RMS voltage rather than peak voltage.
Electronics	RMS noise voltage	Characterizes random electrical noise whose instantaneous mean may be nearly zero. RMS noise gives a useful measure of noise amplitude and is directly related to noise power.
RF engineering	RMS RF voltage/current	Used to relate modulated or irregular RF waveforms to average delivered power, particularly when the waveform cannot be described adequately by a single peak amplitude.
Signal processing	RMS signal amplitude	Gives a scalar measure of the magnitude or energy content of a sampled waveform: $x_{\text{RMS}} = \sqrt{\frac{1}{N} \sum_{n=1}^N x_n^2}$.
Audio engineering	RMS audio amplitude	Describes the effective amplitude of an audio waveform and is generally more representative of sustained signal level than instantaneous peak amplitude.
Acoustics	RMS particle velocity	Describes the oscillatory velocity of air particles in a sound field. It is used with sound pressure to calculate acoustic intensity and acoustic impedance.
Mechanical engineering	RMS vibration displacement	Measures the typical magnitude of an oscillating displacement about an equilibrium position without positive and negative excursions canceling.
Mechanical engineering	RMS vibration velocity	Widely used in rotating-machinery condition monitoring because vibration velocity RMS correlates well with overall machine vibration severity.
Mechanical engineering	RMS vibration acceleration	Common accelerometer-based measure of vibration strength, particularly for high-frequency vibration, impacts, bearing faults, and machinery diagnostics.
Structural engineering	RMS structural displacement	Describes typical motion of buildings, bridges, towers, or other structures under wind, traffic, machinery, or other time-varying excitation.
Structural engineering	RMS structural acceleration	Used to quantify vibration experienced by structures and occupants, including human-comfort assessment in tall buildings and floors.

Domain	Quantity	Description
Vehicle engineering	RMS acceleration	Used to characterize vehicle vibration, ride quality, and occupant exposure to whole-body vibration over time.
Aerospace engineering	RMS vibration acceleration	Used in spacecraft, launch-vehicle, and aircraft vibration testing to characterize random vibration environments and qualification test levels.
Aerospace engineering	RMS pointing error	Measures the typical angular deviation of a spacecraft, telescope, antenna, or guidance system from its desired pointing direction.
Control systems	RMS tracking error	Measures the typical difference between a commanded trajectory and the actual system response over an interval.
Robotics	RMS position error	Summarizes robot position or trajectory accuracy by calculating the RMS difference between commanded and measured positions.
Physics—kinetic theory	RMS molecular speed	Characteristic speed of molecules in a gas, $v_{\text{RMS}} = \sqrt{\langle v^2 \rangle}$. For an ideal gas, $v_{\text{RMS}} = \sqrt{\frac{3kT}{m}}$, directly connecting RMS speed to molecular kinetic energy.
Statistical mechanics	RMS velocity	Used for particles whose vector-average velocity may be zero even though their kinetic energy is nonzero, since kinetic energy depends on v^2 .
Condensed-matter physics	RMS atomic displacement	Describes thermal vibration of atoms around their equilibrium lattice positions and is related to temperature and crystal vibration behavior.
Plasma physics	RMS particle velocity	Used to characterize velocity distributions, thermal motion, fluctuations, and particle energies in plasmas.
Fluid mechanics	RMS velocity fluctuation	Measures the magnitude of turbulent fluctuations about the mean flow. If $u(t) = \bar{u} + u'(t)$, then $u'_{\text{RMS}} = \sqrt{\langle u'^2 \rangle}$.
Turbulence	RMS turbulence intensity	Turbulence strength is often expressed using RMS velocity fluctuations relative to mean velocity, for example $I = u'_{\text{RMS}}/\bar{u}$.
Aerodynamics	RMS pressure fluctuation	Quantifies unsteady pressure caused by turbulence, separated flow, vortex shedding, or aerodynamic loading.
Meteorology	RMS wind-speed fluctuation	Characterizes turbulent or gusting components of wind about the mean wind velocity.

Domain	Quantity	Description
Atmospheric science	RMS temperature fluctuation	Used to quantify fluctuating temperature components in atmospheric turbulence and boundary-layer studies.
Oceanography	RMS surface displacement	Characterizes irregular ocean-wave motion where instantaneous surface elevation varies above and below the mean level.
Oceanography	RMS wave velocity	Describes the effective magnitude of oscillatory water-particle motion associated with waves and currents.
Seismology	RMS ground acceleration	Summarizes the magnitude of time-varying ground acceleration produced by earthquakes, machinery, or ambient seismic noise.
Seismology	RMS ground velocity	Used to characterize earthquake and ambient ground motion while avoiding cancellation between motion in opposite directions.
Geophysics	RMS field anomaly	Used to describe the typical magnitude of variations in measured gravitational, magnetic, seismic, or other geophysical fields relative to a model or mean.
Optics	RMS wavefront error	Measures deviations of an optical wavefront from its ideal shape. RMS wavefront error is a standard measure of optical-system quality.
Optical engineering	RMS spot radius	Characterizes the spread of rays or light intensity around an image point and is used as a measure of imaging performance.
Laser engineering	RMS intensity fluctuation	Used to quantify fluctuations in laser output power or optical intensity, including amplitude noise and source stability.
Surface metrology	RMS surface roughness	The RMS height deviation of a surface from its mean plane, commonly written $R_q = \sqrt{\frac{1}{L} \int_0^L z^2(x) dx}$. It weights large surface deviations more strongly than arithmetic-average roughness.
Manufacturing	RMS dimensional error	Provides an aggregate measure of deviations between manufactured dimensions or surface coordinates and their nominal values.
Statistics	Standard deviation	Standard deviation is the RMS deviation from the mean: $\sigma = \sqrt{\langle (x - \mu)^2 \rangle}$. For a zero-mean quantity, standard deviation and RMS value are identical.
Statistics	Root-mean-square deviation	Measures the typical magnitude of deviations between observed values and a reference or expected value.

Domain	Quantity	Description
Regression	Root-mean-square error (RMSE)	Common model-quality metric, $\text{RMSE} = \sqrt{\frac{1}{N} \sum_{i=1}^N (y_i - \hat{y}_i)^2}$, which weights large prediction errors more strongly than small ones.
Machine learning	RMSE	Used as a loss or evaluation metric for regression models when prediction error is naturally measured in the same units as the predicted quantity.
Numerical analysis	RMS numerical error	Summarizes errors between a numerical approximation and an exact, reference, or higher-accuracy solution over many points.
Computational science	RMS residual	Measures the overall magnitude of residual errors remaining in a numerical solution, iterative solver, or discretized differential equation.
Computational fluid dynamics	RMS residual	Used as a convergence measure for quantities such as continuity, momentum, and energy equations during iterative CFD calculations.
Finite-element analysis	RMS response	Used for structures exposed to random or broadband excitation, including random-vibration and power-spectral-density analyses.
Image processing	RMS pixel error	Measures the difference between two images by taking the RMS of corresponding pixel-value differences.
Imaging	RMS image noise	Characterizes random variations of pixel intensity about the expected signal level.
Computer graphics	RMS rendering error	Can quantify differences between a computed image and a high-quality reference image when evaluating rendering algorithms.
Communications	RMS error-vector magnitude	Characterizes modulation quality by measuring the RMS distance between received constellation points and their ideal locations.
Digital communications	RMS phase error	Measures the typical phase deviation of a received or generated carrier or modulated signal from its ideal value.
Oscillators and timing	RMS jitter	Timing jitter is commonly described by the RMS variation of signal-edge timing relative to an ideal or recovered clock.
Frequency standards	RMS phase fluctuation	Used to characterize short-term phase instability and noise in oscillators, clocks, and frequency references.
Navigation	RMS position error	Measures the typical distance between estimated and true positions over repeated measurements.

Domain	Quantity	Description
GPS/GNSS	RMS ranging error	Describes the statistical magnitude of pseudorange or position errors caused by receiver noise, propagation effects, clocks, and geometry.
Surveying	RMS coordinate error	Used to summarize errors in measured or adjusted coordinates relative to reference positions.
Geodesy	RMS model residual	Describes the typical discrepancy between observed positions, elevations, gravity measurements, or other geodetic quantities and a fitted model.
Astronomy	RMS photometric variation	Quantifies fluctuations in measured stellar brightness and is used in variability studies, instrument characterization, and exoplanet observations.
Astronomy	RMS astrometric error	Describes the typical positional error of measured celestial objects relative to reference coordinates.
Radio astronomy	RMS noise	The RMS fluctuation of receiver or sky-noise measurements sets the noise floor and determines the detectability of weak astronomical signals.
Spectroscopy	RMS spectral noise	Measures fluctuations in spectral intensity about a baseline or fitted spectrum and is often used in estimating signal-to-noise ratio.
Medical imaging	RMS image noise	Used to characterize fluctuations in reconstructed CT, MRI, ultrasound, or other medical images.
Biomedical engineering	RMS EMG amplitude	The RMS amplitude of an electromyography signal provides a measure of muscle activation over a selected time interval.
Biomechanics	RMS acceleration	Used to quantify body-segment motion, gait variability, tremor, impact exposure, or movement intensity from accelerometer data.
Occupational health	RMS vibration exposure	Used in assessing worker exposure to hand-arm or whole-body mechanical vibration over specified time intervals.
Earthquake engineering	RMS response	Used to summarize structural displacement, velocity, or acceleration under earthquake excitation and other random dynamic loads.
Civil engineering	RMS pavement roughness or response	Can characterize variations in road profiles or vehicle responses when evaluating pavement quality and ride comfort.

Domain	Quantity	Description
Hydrology	RMS model error	Used to evaluate differences between modeled and measured quantities such as streamflow, water level, or precipitation.
Climate science	RMS climate-model error	Summarizes spatial or temporal differences between model predictions and observations for quantities such as temperature or precipitation.
Remote sensing	RMS geolocation error	Measures typical positional discrepancy between remotely sensed data and known reference locations.
Photogrammetry	RMS reprojection error	Measures the discrepancy between observed image points and points projected from a fitted camera or geometric model.
Computer vision	RMS reprojection error	Common measure of camera-calibration quality based on differences between measured and predicted image coordinates.
Metrology	RMS measurement error	Describes the effective magnitude of repeated measurement deviations from a reference value or fitted model.
Frequency-domain analysis	RMS spectral amplitude	RMS values can be derived from spectral components or power spectral density, allowing total signal or noise level to be obtained by summing squared contributions.
Random-process theory	RMS value of a stochastic process	For a random process $X(t)$, the RMS magnitude is $X_{\text{RMS}} = \sqrt{E[X^2]}$. It provides a natural measure of the strength of random quantities that may have zero mean.
Probability theory	RMS random variable	The quantity $\sqrt{E[X^2]}$ is the RMS magnitude of a random variable and combines contributions from both its mean and its variance: $X_{\text{RMS}}^2 = \mu^2 + \sigma^2$.
Error analysis	RMS error	General measure for signed errors where positive and negative errors should not cancel. Squaring makes all contributions positive and places greater weight on large errors.

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1. To the mathematical folk that just twitched: yes, a proper square root operation includes the $\pm$ prefix. Remember that the V_{eff} definition involving power *always* involves squaring first. Therefore there is no change if you take the negative value, the same power is dissipated in the resistor. By convention, we take the positive value to reduce the negativity in our lives.